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# Multiple solitary-waves due to second harmonic generation in quadratic media

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## Abstract

A detailed mathematical analysis is undertaken of solitary wave solutions of a system of coupled NLS equations describing second harmonic generation in optical materials with  $\chi^{(2)}$  nonlinearity. The so called bright-bright case is studied exclusively. The system depends on a single dimensionless parameter  $\alpha$  which includes both wave and material properties. Using variational methods, the first rigorous mathematical proof is given that at least one solitary wave exists for all positive  $\alpha$ . Recently bound states (multi-pulsed solitary waves) have been found numerically. Using numerical continuation, the region of existence of these solutions is revealed to be  $\alpha \in (0, 1)$  and the bifurcations occurring at the two extremes of this interval are uncovered.

## 1 Introduction

This paper concerns the application of mathematical existence theory and numerical continuation to a certain problem in nonlinear optics. The approach taken here is indicative of a method which is likely to be more generally applicable in this field.

In recent years there has been a resurgence of interest in solitary waves occurring in optical media with quadratic or  $\chi^{(2)}$  nonlinearity. See [1, 2, 3, 4] and references therein. For such materials the combination of self-focusing and second-harmonic generation is of importance for example in parametric waveguides and in ultrafast all-optical switching.

The phenomena of interest in 2 dimensions (either space+time or two space dimensions) can be adequately described by a system of two coupled nonlinear Schrödinger (NLS) equations written in the general form

$$\begin{aligned} i \frac{\partial W}{\partial \zeta} + r \frac{\partial^2 W}{\partial t^2} - W + W^* V &= 0, \\ i \sigma \frac{\partial V}{\partial \zeta} + s \frac{\partial^2 V}{\partial t^2} - \alpha V + \frac{1}{2} W^2 &= 0, \end{aligned} \tag{1.1}$$

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with  $r, s = \pm 1$ . This is obtained from the basic  $\chi^{(2)}$  second-harmonic generation equations after non-dimensionalisation, the use of moving frame coordinates, and the insertion of an ansatz of the form  $W(\zeta, t) \exp(i\phi_1(\zeta, t))$ ,  $V(\zeta, t) \exp(2i\phi_2(\zeta, t))$ . See e.g. [2] for a derivation. Note that in the spatial case, system (1.1) is valid when no walk-off is present. The complex variables  $W(\zeta, t)$  and  $V(\zeta, t)$  represent respectively the first and second harmonics of an optical pulse. The signs of  $r$  and  $s$  are determined by the signs of the dispersions (for the temporal case), and  $\sigma$  measures the ratios of the dispersions. The single dimensionless parameter  $\alpha$  is a combination of  $\sigma$  and the wavevector mismatch.

Stationary solutions of (1.1) (in the temporal case these represent travelling waves where the primary and second harmonic travel with the same group velocity) may be obtained upon setting  $W = w(t)$ ,  $V = v(t)$ . This yields the ordinary differential equations

$$\begin{aligned} rw'' - w + wv &= 0 \\ sv'' - \alpha v + \frac{1}{2}w^2 &= 0 \end{aligned} \tag{1.2}$$

in which  $w(t)$  and  $v(t)$  may be taken to be real [2]. The system (1.2) is Hamiltonian with total energy

$$H = r\frac{w'^2}{2} + s\frac{v'^2}{2} + \frac{1}{2}(w^2v - \alpha v^2 - w^2).$$

In this article we consider only the ‘bright-bright’ case where  $r = s = +1$ . For such a system we shall be interested in homoclinic solutions to the origin, which describe pulses of the PDE system (1.1). The eigenvalues of the linearization at the origin are  $\pm\sqrt{\alpha}$  and  $\pm 1$ . At  $\alpha = 1$  the eigenvalues form two semi-simple pairs of eigenvalues  $\pm 1$ . At this single value of  $\alpha$ , (1.2) possesses the exact homoclinic solution

$$w_s^\pm = (3/\sqrt{2})\text{sech}^2(t/2), \quad v_s^\pm = (3/2)\text{sech}^2(t/2),$$

found by a number of authors, e.g. [5, 2].

It was demonstrated by Buryak & Kivshar [2], using numerical shooting, that a continuous branch of primary homoclinic solutions can be traced for all  $\alpha > 0$ , which contains the exact solution  $(w_s^+, v_s^+)$  at  $\alpha = 1$ . In the limit  $\alpha \rightarrow \infty$ , the solution on this branch asymptotes to

$$w \approx 2\sqrt{\alpha}\text{sech}(t), \quad v \approx 2\text{sech}^2(t),$$

with the ratio of the amplitudes of  $w$  to  $v$  being a monotonically increasing function of  $w$ . In [3] a plausible topological argument for the existence of such a branch is given, based on the numerically computed acceleration field of the mechanical equivalent of (1.2).

Recently He *et al.* [3] (see also [4]) have found numerically the existence of multi-pulse homoclinic solutions which are like multiple copies of the primary solution (e.g. see Figure 1 below). However, it is not clear from these cited works for precisely what range of  $\alpha$ -values such solutions exist. Moreover Haelterman *et al* [4] present numerical simulations of the PDEs (1.1) that suggest that while the primary solution is stable, the multi-pulse solutions are unstable. The mode of instability depends on the size of  $\alpha$ . Nonetheless, the existence of the multi-pulse solutions is of interest in describing the interaction between primary pulses which is fundamental for optical switching devices.

The purpose of this paper is twofold. First, in Section 2, we give the first rigorous analytic proof there is at least one homoclinic solution to (1.2) for all positive  $\alpha$ . Second, in Section 3, we provide strong numerical evidence that there are infinitely many multi-pulse homoclinic solutions for all  $\alpha \in (0, 1)$ . In particular, we shall numerically probe what happens to these solutions as  $\alpha \rightarrow 0$  and  $\alpha \rightarrow 1$ . Finally, in Section 4 we draw conclusions and highlight some open analytical questions.

## 2 Existence of one homoclinic solution

In this section it will be shown that for the system

$$w'' - w + wv = 0 \quad (2.1)$$

$$v'' - \alpha v + \frac{1}{2}w^2 = 0$$

there exists at least one solution homoclinic to the origin (corresponding physically to a pulse), for all positive  $\alpha$ . The method will be of a variational nature, using the Mountain Pass Lemma and the Concentration–Compactness ideas to be found in [6, 7, 8]. This approach has already been applied to various fourth-order wave equations; see, for example [12, 9].

**Theorem 2.1** *For every  $\alpha > 0$ , the system (2.1) has at least one non-trivial solution which is homoclinic to the origin.*

Consider the functional  $I \in C^\infty(H^1(\mathbb{R}, \mathbb{R}^2), \mathbb{R})$  defined by

$$I(u) = \int_{\mathbb{R}} \left( \frac{1}{2}(w'^2 + v'^2) + \frac{1}{2}(w^2 + \alpha v^2) - \frac{1}{2}vw^2 \right) dt \quad (2.2)$$

for  $u = (w, v) \in H^1(\mathbb{R}, \mathbb{R}^2)$ . Its Fréchet derivative is

$$I'(u)\Phi = \int_{\mathbb{R}} \left( w'\phi' + v'\psi' + w\phi + \alpha v\psi - vw\phi - \frac{1}{2}w^2\psi \right) dt \quad (2.3)$$

for  $\Phi = (\phi, \psi) \in H^1(\mathbb{R}, \mathbb{R}^2)$ . Observe that the critical points of  $I$  correspond to weak solutions of the system of differential equations (2.1) which decay to the origin as  $t \rightarrow \pm\infty$ . Moreover, they are in fact classical solutions, see [10] (Chapter 2) or [11].

First we will show via a Mountain Pass Lemma [12, 10] the existence of a sequence  $\{u_n\}$ , with  $\|I'(u_n)\| \rightarrow 0$ .

**Theorem 2.2 (Mountain Pass Lemma)** *Let  $E$  be a real Banach space, and  $I \in C^1(E, \mathbb{R})$ . Suppose (H1) for some  $\omega \in E$ , there exists  $a, \rho > 0$  such that*

$$I|_{\partial B_\rho(\omega)} \geq I(\omega) + a$$

*where  $B_\rho(\omega)$  is a ball of radius  $\rho$  in  $E$  centred at  $\omega$ ,*

*(H2) there exists  $e \in E \setminus \bar{B}_\rho(\omega)$  such that*

$$I(e) \leq I(\omega) ,$$

*then there exists a sequence  $\{u_n\} \subseteq E$  such that*

$$I(u_n) \rightarrow c \text{ and } \|I'(u_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (2.4)$$

*where*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u) \quad (2.5)$$

*with  $\Gamma = \{g \in C([0, 1]) \mid g(0) = \omega, g(1) = e\}$ .*

Here we take  $\omega = 0$ ,  $I(\omega) = 0$ ,  $E = H^1(\mathbb{R}, \mathbb{R}^2)$ . We need to verify (H1) and (H2). Note that for (H1) it suffices to show 0 is a local minimum of  $I$ , while for (H2) it is enough to prove that  $I(Au) \rightarrow -\infty$  as  $A \rightarrow \infty$  for some fixed  $u$ .

Consider the following norm induced by the functional  $I$ :

$$\|u\|_I^2 = \int_{\mathbb{R}} (w'^2 + v'^2 + w^2 + \alpha v^2) dt \quad (2.6)$$

for  $u \in H^1(\mathbb{R}, \mathbb{R}^2)$ .

**Remark 2.1** For fixed  $\alpha > 0$ ,  $C_1\|u\|_E \leq \|u\|_I \leq C_2\|u\|_E$  for some  $C_1, C_2 > 0$  where

$$\|u\|_E = \|(w, v)\|_{H^1} = \int (w'^2 + v'^2 + w^2 + v^2) dt$$

i.e. the norms  $\|\cdot\|_I$  and  $\|\cdot\|_E$  are equivalent.

**Proof** Comparing (2.6) and the definition of  $\|\cdot\|_E$  above, it is easy to see that for  $\alpha > 1$ , we can take  $C_1 = 1$ ,  $C_2 = \alpha$ , while for  $\alpha < 1$ , can take  $C_1 = \alpha$ ,  $C_2 = 1$ . ■

**Lemma 2.1**  $\|u\|_\infty \leq 2\|u\|_E$  where

$$\|u\|_\infty = \|(w, v)\|_\infty = \sup_{t \in \mathbb{R}} |w(t)| + \sup_{t \in \mathbb{R}} |v(t)|$$

**Proof**  $u \in H^1(\mathbb{R})$  is continuously embedded in  $C_b(\mathbb{R})$ , the space of continuous bounded functions on  $\mathbb{R}$ , with  $\|u\|_\infty \leq C\|u\|_E$  for some  $C > 0$ , by the Sobolev Embedding Theorem (see, for example [13, 14]). This is in fact all that is needed, but for completeness we will show  $\|u\|_\infty \leq 2\|u\|_E$ , following the approach taken in [12]. We first consider any  $t \in (0, 1)$  and write, using the Fundamental Theorem of Calculus:

$$w^2(t) = w^2(a) + \int_a^t (2w(s)w'(s)) ds$$

where  $a \in (0, 1)$  is chosen so that

$$w^2(a) = \int_0^1 w^2(s) ds$$

by the Mean Value Theorem with  $f(t) = \int_0^t w^2(s) ds$ . Now

$$\begin{aligned} \int_a^t w(s)w'(s) ds &\leq \|w\|_{L^2(0,1)} \|w'\|_{L^2(0,1)} \\ &\leq \frac{1}{2} (\|w\|_{L^2(0,1)}^2 + \|w'\|_{L^2(0,1)}^2) \end{aligned}$$

hence

$$\begin{aligned} w^2(t) &\leq \int_0^1 w^2(s) ds + \|w\|_{L^2(0,1)}^2 + \|w'\|_{L^2(0,1)}^2 \\ &\leq 2\|w\|_{L^2(0,1)}^2 + \|w'\|_{L^2(0,1)}^2 \end{aligned}$$

so

$$\sup_{t \in (0,1)} |w(t)|^2 \leq 2 (\|w\|_{L^2(0,1)}^2 + \|w'\|_{L^2(0,1)}^2)$$

We can do this for all intervals  $(n, n + 1)$ ,  $n = 0, 1, 2, \dots$ , and similarly for  $v(t)$ . Putting these results together, we obtain:

$$\begin{aligned} \sup_{t \in \mathbb{R}} |w(t)|^2 + \sup_{t \in \mathbb{R}} |v(t)|^2 &\leq 2 \left( \|w\|_{L^2(\mathbb{R})}^2 + \|w'\|_{L^2(\mathbb{R})}^2 + \|v\|_{L^2(\mathbb{R})}^2 + \|v'\|_{L^2(\mathbb{R})}^2 \right) \\ &\leq 2 \|(w, v)\|_{H^1(\mathbb{R})}^2 \end{aligned}$$

whence  $\|(w, v)\|_\infty \leq 2\|(w, v)\|_E$  as claimed. ■

**Corollary 2.1**  $|\int_{\mathbb{R}} v(t)w^2(t) dt| \leq 2 \|(w, v)\|_E^3$

**Proof**

$$\begin{aligned} \left| \int v w^2 dt \right| &\leq \int |v w^2| dt \\ &\leq \sup_{t \in \mathbb{R}} |v(t)| \int w^2(t) dt \\ &\leq \|(w, v)\|_\infty \|(w, v)\|_E^2 \\ &\leq 2 \|(w, v)\|_E^3 \end{aligned}$$

by the previous lemma. ■

Note that

$$I((w, v)) = \frac{1}{2} \|(w, v)\|_I^2 - \frac{1}{2} \int_{\mathbb{R}} v(t)w^2(t) dt \quad (2.7)$$

Now we verify (H1) and (H2).

**Lemma 2.2** *The functional  $I$  given in (2.2) has a strict local minimum at 0 in the function space  $E = H^1(\mathbb{R}, \mathbb{R}^2)$ ; condition (H1) holds.*

**Proof** From Corollary 2.1, we have

$$-\int_{\mathbb{R}} v w^2 dt \geq -2\|u\|_E^3$$

where  $u = (w, v)$ . Therefore, by equation (2.7) and Remark 2.1,

$$\begin{aligned} I(u) &\geq \frac{1}{2} \left( \frac{1}{C_2} \|u\|_E^2 - 2\|u\|_E^3 \right) \\ &\geq \frac{1}{2C_2} \|u\|_E^2 (1 - 2C_2\|u\|_E) \end{aligned}$$

The graph of  $\frac{1}{C} \rho^2 (1 - C\rho)$  is strictly positive for  $\rho \in (0, \frac{1}{C})$ . Take, for instance,  $\rho = \frac{2}{3C}$ , then for  $u \in \partial B_\rho$ , i.e.  $\|u\|_E = \rho$ , have  $I(u) \geq \frac{1}{C} \left( \frac{2}{3C} \right)^2 \left( 1 - C \frac{2}{3C} \right) = \frac{4}{27C^3} = a > 0$ . ■

**Lemma 2.3** *Take  $I$  as given in (2.2). Let  $u_0 = (w_0, v_0)$  be a fixed element in  $E$  such that  $\int_{\mathbb{R}} v_0 w_0^2 dt > 0$ , then  $I(Au_0) \rightarrow -\infty$  as  $A \rightarrow +\infty$ .*

**Proof**

$$I(Au_0) = \int_{\mathbb{R}} \left( \frac{A^2}{2}(w_0^2 + \alpha v_0^2) + \frac{A^2}{2}(w_0'^2 + v_0'^2) - \frac{A^3}{2}v_0 w_0^2 \right) dt$$

hence

$$\frac{I(Au_0)}{A^2} = \frac{1}{2}\|u_0\|_I^2 - \frac{A}{2} \int_{\mathbb{R}} v_0 w_0^2 dt \quad (2.8)$$

Choose  $u_0 = (v_0, w_0) \in E$  with  $\int_{\mathbb{R}} v_0 w_0^2 dt > 0$ , then (2.8) gives:

$$\lim_{A \rightarrow \infty} \left( \frac{I(Au_0)}{A^2} \right) = -\infty$$

therefore certainly  $I(Au_0) \rightarrow -\infty$  as  $A \rightarrow +\infty$ . ■

Let  $e = Au_0$  with  $A$  sufficiently large, then (H2) in the statement of Theorem 2.2 holds.

Theorem 2.2 now gives us a sequence  $\{u_n\} \subseteq H^1(\mathbb{R}, \mathbb{R}^2)$  satisfying the properties (2.4). In order to prove the existence of a nontrivial critical point of  $I$  (corresponding to a homoclinic solution of (2.1)), we still need to:

- Pick a ‘limit’ function  $\hat{u}$  of  $u_n$  (in some appropriate sense).
- Show that  $\hat{u}$  is non-trivial.
- Show that  $I'(\hat{u})\Phi = 0$  for all  $\Phi = (\phi, \psi) \in H^1(\mathbb{R}, \mathbb{R}^2)$ .

The following lemmata will be useful:

**Lemma 2.4** *Take  $I$  as given in (2.2). Any sequence  $\{u_n\} \subseteq H^1(\mathbb{R}, \mathbb{R}^2)$  satisfying (2.4) must be bounded.*

**Proof** Suppose that  $\{u_n\} = \{(w_n, v_n)\}$  satisfies (2.4), but  $\|u_n\| \rightarrow \infty$  as  $n \rightarrow \infty$  (where  $\|\cdot\|$  can be either  $\|\cdot\|_I$  or  $\|\cdot\|_{H^1}$ ). It follows that:

$$\frac{I(u_n)}{\|u_n\|^2} \rightarrow 0 \quad (2.9)$$

and also

$$\frac{I'(u_n)u_n}{\|u_n\|^2} \rightarrow 0 \quad (2.10)$$

as  $n \rightarrow \infty$ . Now

$$I(u_n) = \frac{1}{2} \int_{\mathbb{R}} (w_n'^2 + v_n'^2 + w_n^2 + \alpha v_n^2 - v_n w_n^2) dt$$

$$I'(u_n)u_n = \int_{\mathbb{R}} \left( w_n'^2 + v_n'^2 + w_n^2 + \alpha v_n^2 - v_n w_n^2 - \frac{1}{2}v_n w_n^2 \right) dt$$

from (2.3) with  $(\phi, \psi) = (w_n, v_n)$ . Therefore:

$$2I(u_n) - I'(u_n)u_n = \frac{1}{2} \int_{\mathbb{R}} v_n w_n^2 dt \quad (2.11)$$

and hence

$$0 = \lim_{n \rightarrow \infty} \frac{\frac{1}{2} \int v_n w_n^2 dt}{\|u_n\|^2} \quad (2.12)$$

by (2.9) and (2.10). We can also write

$$I'(u_n)u_n = \|u_n\|_I^2 - 3 \left( \frac{1}{2} \int v_n w_n^2 dt \right)$$

Dividing through by  $\|u_n\|^2$  and letting  $n \rightarrow \infty$  gives

$$0 = 1 - 3(0) = 1$$

which is a contradiction. Therefore  $\{u_n\}$  must be bounded. ■

**Lemma 2.5** *Let  $\{u_n\} = \{(w_n, v_n)\} \subseteq H^1(\mathbb{R}, \mathbb{R}^2)$  be a sequence satisfying (2.4), where  $I$  is given by (2.2), then*

$$\sup_{t \in \mathbb{R}} |v_n(t)| > a$$

*for all  $n$  sufficiently large, for some  $a > 0$ .*

**Proof** Taking the limit  $n \rightarrow \infty$  of (2.11), and using the boundedness proved in Lemma 2.4, we obtain

$$2c - 0 = \lim_{n \rightarrow \infty} \left( \frac{1}{2} \int v_n w_n^2 dt \right)$$

where  $c$  given in (2.5) is positive. So

$$4c \leq \lim_{n \rightarrow \infty} \int |v_n| w_n^2 dt$$

which implies that for  $n$  sufficiently large,

$$\begin{aligned} 2c &< \int |v_n| w_n^2 dt \\ &\leq \sup_{t \in \mathbb{R}} |v_n(t)| \|w_n\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \\ &\leq \sup_{t \in \mathbb{R}} |v_n(t)| \|u_n\|_E^2 \end{aligned}$$

Now  $\|u_n\|_E$  is bounded – by  $B$ , say – hence taking  $a = \frac{2c}{B^2}$  gives us the result. ■

Without loss of generality assume  $\{u_n(\cdot)\} = \{(w_n(\cdot), v_n(\cdot))\}$  is such that the statement in Lemma 2.5 holds for every  $n$ . Then, for each  $n$  choose  $t_n$  such that

$$|u_n(t_n)| \geq |v_n(t_n)| \geq a$$

where  $a > 0$  is the lower bound given in Lemma 2.5. Define

$$U_n(t) \equiv (W_n(t), V_n(t)) = (w_n(t_n + t), v_n(t_n + t)) \equiv u_n(t_n + t) \quad (2.13)$$

for all  $t \in \mathbb{R}$ . Observe that  $I(U_n) = I(u_n)$  and  $\{U_n\}$  is bounded in  $H^1(\mathbb{R}, \mathbb{R}^2)$ . This implies that there exists a subsequence of  $\{U_n\}$  (which we shall also denote by  $\{U_n\}$ ) that converges weakly to a unique limit function  $\hat{U} \in H^1(\mathbb{R}, \mathbb{R}^2)$ , as  $H^1(\mathbb{R}, \mathbb{R}^2)$  is a Hilbert space. That is:

$$\langle U_n, \Phi \rangle \rightarrow \langle \hat{U}, \Phi \rangle \text{ for all } \Phi \in H^1(\mathbb{R}, \mathbb{R}^2) \quad (2.14)$$



(where the inner product can be that associated with either of  $\|\cdot\|_I$ ,  $\|\cdot\|_{H^1}$ ) or:

$$J(U_n) \rightarrow J(\hat{U}) \text{ for all } J \in (H^1(\mathbb{R}, \mathbb{R}^2))^* \quad (2.15)$$

Taking  $J$  to be evaluation at  $t = 0$  gives

$$|U_n(0)| = |J(U_n)| \rightarrow |J(\hat{U})| = |\hat{U}(0)|$$

We know that  $|U_n(0)| = |u_n(t_n)| \geq a$  for all  $n$ , hence

$$|\hat{U}(0)| \geq a > 0$$

and so  $\hat{U}$  is non-trivial.

It remains to show that

$$I'(\hat{U})\Phi = 0 \text{ for all } \Phi = (\phi, \psi) \in H^1(\mathbb{R}, \mathbb{R}^2) \quad (2.16)$$

From (2.3),

$$\begin{aligned} I'(U_n)\Phi - I'(\hat{U})\Phi &= \langle U_n, \Phi \rangle - \langle \hat{U}, \Phi \rangle \\ &\quad - \int_{\mathbb{R}} \left( W_n V_n \phi + \frac{1}{2} W_n^2 \psi \right) dt + \int_{\mathbb{R}} \left( \hat{W} \hat{V} \phi + \frac{1}{2} \hat{W}^2 \psi \right) dt \end{aligned}$$

where

$$\langle (f_1, f_2), (g_1, g_2) \rangle = \int_{\mathbb{R}} (f_1' g_1' + f_2' g_2' + f_1 g_1 + \alpha f_2 g_2) dt$$

is the inner product associated with  $\|\cdot\|_I$ . From (2.14),

$$\lim_{n \rightarrow \infty} (\langle U_n, \Phi \rangle - \langle \hat{U}, \Phi \rangle) = 0$$

Also, as  $n \rightarrow \infty$ ,

$$|I'(U_n)\Phi| \leq \|I'(U_n)\| \|\Phi\|_{H^1} \rightarrow 0$$

So (2.16) is proved if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left( W_n V_n \phi + \frac{1}{2} W_n^2 \psi \right) dt - \int_{\mathbb{R}} \left( \hat{W} \hat{V} \phi + \frac{1}{2} \hat{W}^2 \psi \right) dt = 0$$

We will show

**Lemma 2.6** *If  $(W_n, V_n) \rightarrow (\hat{W}, \hat{V})$  weakly in  $H^1$  as  $n \rightarrow \infty$ , where  $W_n, V_n, \hat{W}, \hat{V}$  are in  $H^1(\mathbb{R}, \mathbb{R})$ , and  $\phi \in H^1(\mathbb{R}, \mathbb{R})$  is arbitrary, then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} (W_n V_n \phi - \hat{W} \hat{V} \phi) dt = 0 \quad (2.17)$$

**Remark 2.2** *Taking  $V_n = W_n$ ,  $\hat{V} = \hat{W}$ ,  $\phi = \psi$  gives*

$$\lim_{n \rightarrow \infty} \int \frac{1}{2} (W_n^2 - \hat{W}^2) \psi dt = 0$$

**Proof of lemma** For  $K$  being a compact subset of  $\mathbb{R}$ ,  $H^1(K)$  is compactly embedded in  $C^\alpha(K)$ , for  $\alpha < \frac{1}{2}$  (Sobolev Embedding Theorem [13]). In particular,  $H^1(K, \mathbb{R})$  is compactly embedded in  $C^0(K, \mathbb{R})$ . Now

$$\|(W_n, V_n)\|_{H^1(\mathbb{R}, \mathbb{R}^2)} \leq \tilde{M}$$

for some  $\tilde{M}$  independent of  $n$ , and we have:  $\{W_n\}$  (or a subsequence of it) converges uniformly to  $\hat{W}$  on compact sets  $K \subseteq \mathbb{R}$ ; similarly for  $V_n$  and  $\hat{V}$ . Therefore

$$\begin{aligned} \|W_n V_n - \hat{W} \hat{V}\|_{L^1(\mathbb{R}, \mathbb{R})} &= \int_{\mathbb{R}} |W_n V_n - \hat{W} \hat{V}| dt \\ &\leq \int |W_n V_n| dt + \int |\hat{W} \hat{V}| dt \\ &\leq \frac{1}{2} \int (W_n^2 + V_n^2) dt + \frac{1}{2} \int (\hat{W}^2 + \hat{V}^2) dt \\ &\leq \tilde{M}^2 + \frac{1}{2} \|(\hat{W}, \hat{V})\|_{H^1} \\ &\leq M \quad (\text{independent of } n) \end{aligned}$$

Given any  $\phi \in H^1(\mathbb{R}, \mathbb{R})$  and small  $\varepsilon > 0$ , choose a compact set  $K \subseteq \mathbb{R}$  such that

$$\sup_{t \in \mathbb{R} \setminus K} |\phi(t)| < \frac{\varepsilon}{2M}$$

Then on  $\mathbb{R} \setminus K$ ,

$$\begin{aligned} \left| \int_{\mathbb{R} \setminus K} (W_n V_n - \hat{W} \hat{V}) \phi dt \right| &\leq \sup_{t \in \mathbb{R} \setminus K} |\phi(t)| \|W_n V_n - \hat{W} \hat{V}\|_{L^1(\mathbb{R})} \\ &< \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2} \end{aligned} \tag{2.18}$$

On  $K$ : choose  $N$  such that for all  $n > N$  and all  $t \in K$ ,

$$\begin{aligned} \|W_n(t) - \hat{W}(t)\| &\leq \frac{\varepsilon}{6A \|\hat{V}\|_{L^\infty(K)}} \\ \text{and } \|V_n(t) - \hat{V}(t)\| &\leq \frac{\varepsilon}{6A \|\hat{W}\|_{L^\infty(K)}} \end{aligned}$$

where

$$A = \int_K |\phi(t)| dt < \infty$$

Consider, for all  $n > N$  and  $t \in K$ ,

$$\begin{aligned} |W_n(t) V_n(t) - \hat{W}(t) \hat{V}(t)| &\leq |W_n(t)| |V_n(t) - \hat{V}(t)| + |W_n(t) - \hat{W}(t)| |\hat{V}(t)| \\ &\leq |W_n(t) - \hat{W}(t)| |V_n(t) - \hat{V}(t)| \\ &\quad + |\hat{W}(t)| |V_n(t) - \hat{V}(t)| + |W_n(t) - \hat{W}(t)| |\hat{V}(t)| \\ &\leq \|\hat{W}\| \frac{\varepsilon}{6A \|\hat{W}\|} + \frac{\varepsilon^2}{36A^2 \|\hat{W}\| \|\hat{V}\|} \\ &\quad + \|\hat{V}\| \frac{\varepsilon}{6A \|\hat{V}\|} \end{aligned}$$

(all norms refer to the  $L^\infty(K)$  norm). Since  $\varepsilon$  is small, without loss of generality we can assume  $\varepsilon \leq 6A\|\hat{W}\|\|\hat{V}\|$  (or just enlarge  $K$ ), hence

$$|W_n(t)V_n(t) - \hat{W}(t)\hat{V}(t)| \leq \frac{\varepsilon}{6A} + \frac{\varepsilon}{6A} + \frac{\varepsilon}{6A} = \frac{\varepsilon}{2A}$$

Therefore on  $K$  we have, for sufficiently large  $n$ ,

$$\begin{aligned} \left| \int_K (W_n V_n - \hat{W} \hat{V}) \phi \, dt \right| &\leq \sup_{t \in K} |W_n(t)V_n(t) - \hat{W}(t)\hat{V}(t)| \int_K |\phi(t)| \, dt \\ &\leq A \frac{\varepsilon}{2A} = \frac{\varepsilon}{2} \end{aligned} \tag{2.19}$$

Combining the results (2.18) and (2.19), we obtain the lemma. ■

We can thus conclude that Theorem 2.1 holds.

**Remarks:**

- (a) Due to the invariance of system (2.1) under the symmetry  $(w, v) \mapsto (-w, v)$ , it follows that  $(-\hat{W}(t), \hat{V}(t))$  is also a solution homoclinic to the origin, so in fact we have shown the existence of at least two pulses.
- (b) It is possible to extract more information about the structure of critical points of functional (2.2), but this is not needed for the present paper.

### 3 Numerical results

In this section we present numerical results on homoclinic solutions to the origin of (2.1). First note that this system is invariant under the reversibility

$$R : (w, w', v, v') \mapsto (w, -w', v, -v'), \quad t \mapsto -t$$

and also under the action of  $\mathbb{Z}_2$

$$Z : (w, w', v, v') \mapsto (-w, -w', v, v').$$

Note that  $ZR = RZ$  also defines a reversibility for (1.2). We shall seek solutions which are either  $R$ -reversible or  $ZR$ -reversible, using the methods described in [15]. These methods, which rely heavily on the reversibility, are based on shooting and numerical continuation of two-point boundary value problems. Numerical continuation is performed using the software AUTO [16]. For definiteness we shall only compute solutions for which  $w$  is positive for  $t$  sufficiently small. Given any such solution, its image under  $Z$  will also be a solution.

We begin by fixing  $\alpha = 0.5$ . Figure 1 shows five solutions. The solution in (a) we refer to as the primary solution. It is reversible under  $R$  and is the one which forms a continuous branch with the exact solution  $(w_s^+, v_s^+)$  at  $\alpha = 1$ . The solutions in (b), (d) and (f) are reversible under  $ZR$  and consist of approximately an even number of copies of the primary orbit, with the sign of  $w$ -component alternating. Plots (c) and (e) depict solutions which are  $R$ -reversible and are approximately an odd number of copies of the primary, again with  $w$  alternating in sign. It is reasonable to conjecture from these numerical results that this process of gluing solutions together can be extended to give an infinity of solutions consisting of arbitrarily large numbers of copies of the primary.

We were unable to find any localised solution other than the primary that is everywhere positive. In particular, based on a careful numerical search, we conjecture that positive multiple pulsed solutions

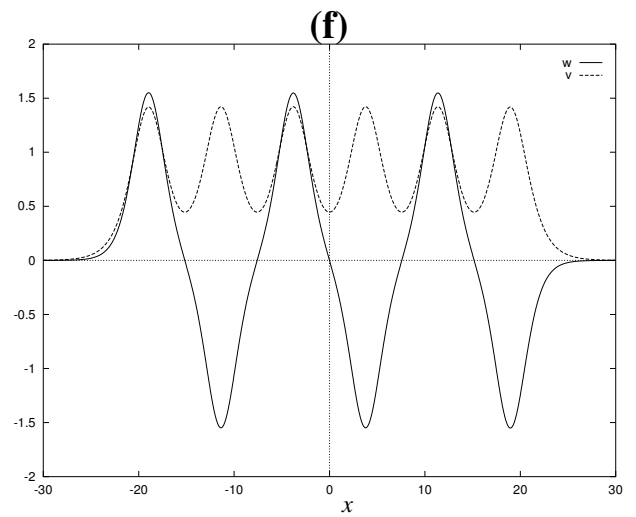
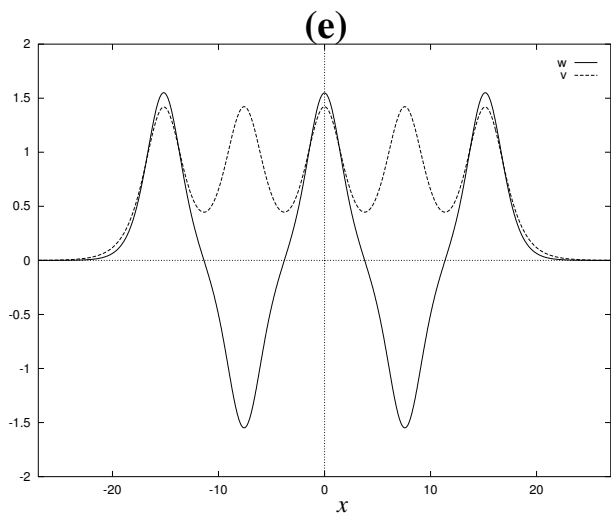
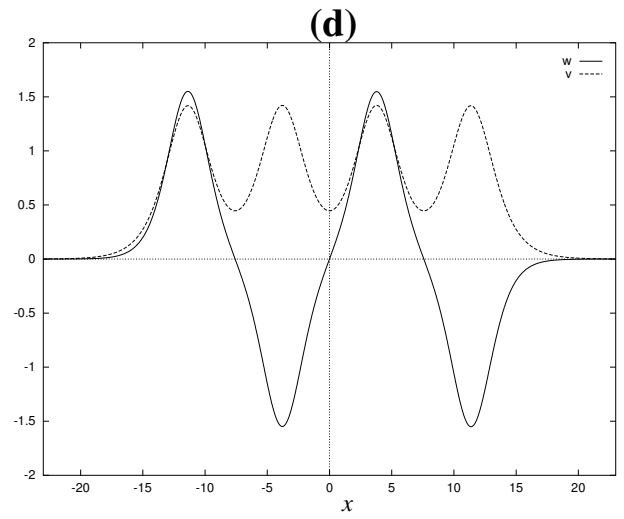
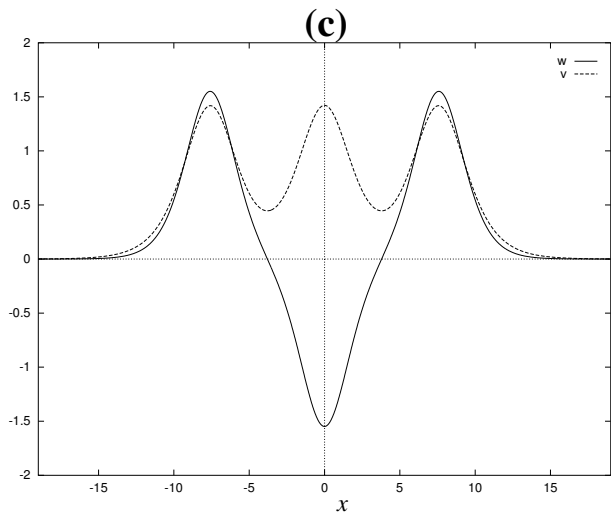
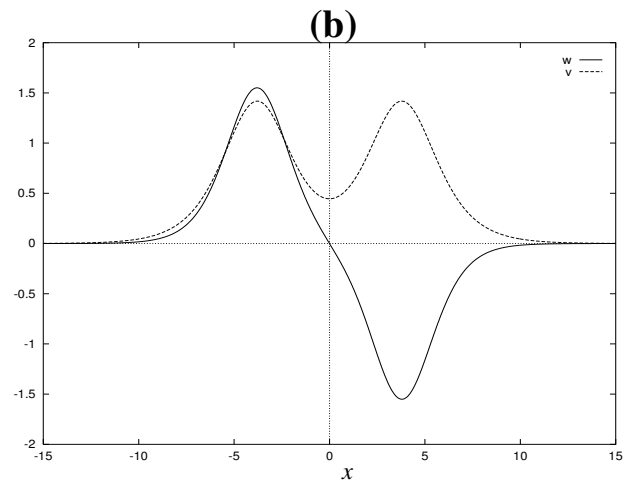
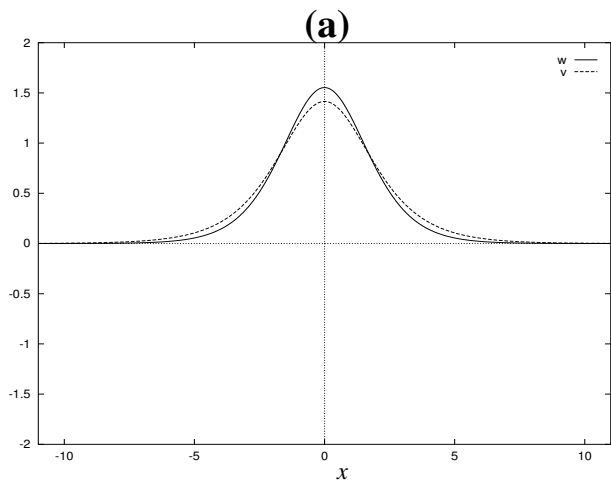


Figure 1: Six homoclinic solutions for  $\alpha = 0.5$ .

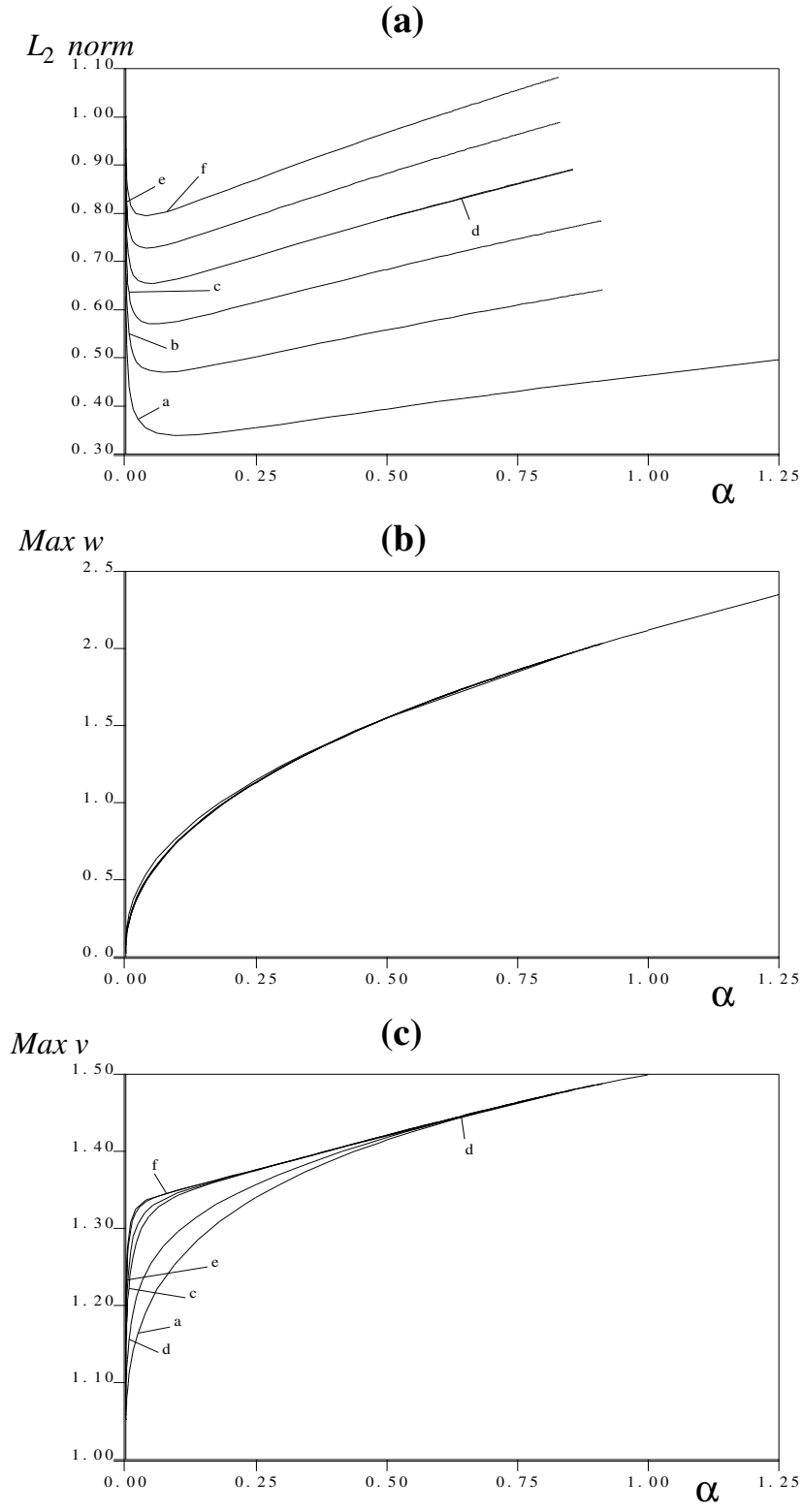


Figure 2: Bifurcation diagram of amplitudes against  $\alpha$ . The letters a–f refer to solutions depicted in the corresponding plots in Figure 1. The ordinate in (a) is the vector  $L_2$  norm of  $(w(t), w'(t), v(t), v'(t))$  scaled by a factor of 100, which is the  $t$ -interval over which all solutions were computed. In (b) all solution branches are approximately overlaid.

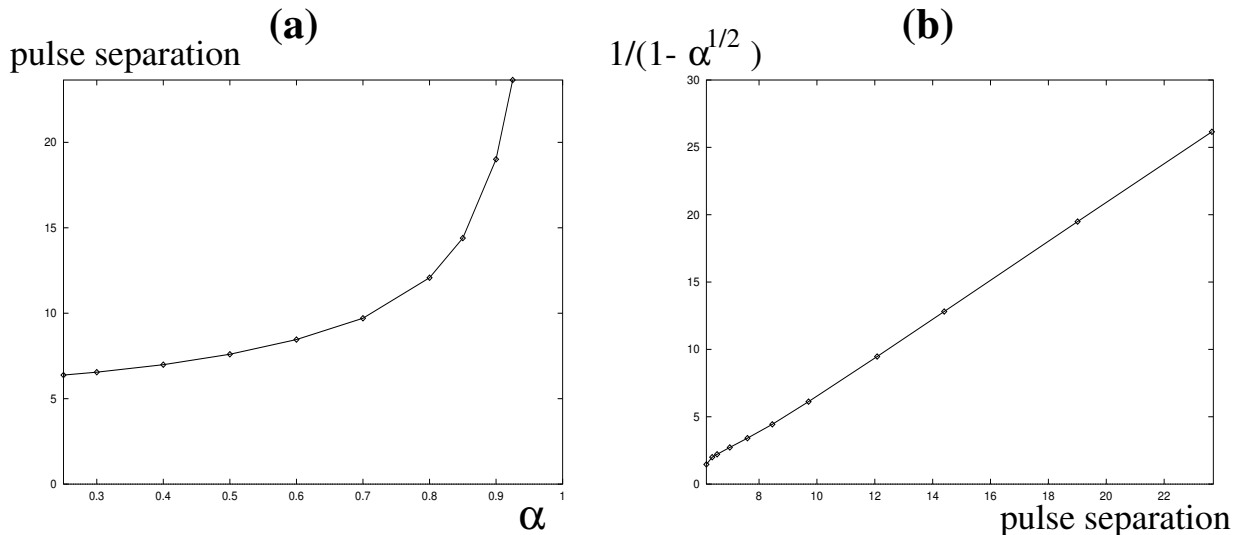


Figure 3: The separation of the two humps of the two-pulse solution; (a) separation plotted against  $\alpha$ ; (b) showing the scaling between separation and  $\alpha$  as  $\alpha \rightarrow 1$ . scale

do not exist for  $\alpha = 0.5$ . Such solutions would be more physically interesting than the multi-pulse solutions in Figure 1(b)–(f) because they represent couplings between ‘bright solitons’ in the first and second harmonics.

Figure 2 depicts the branches of solutions on which those on Figure 1 lie, and Figures 4,5,6 depict solutions at various values of  $\alpha$ . These results strongly suggest that the primary orbit exists for all  $\alpha > 0$  whereas each multiple pulsed solution exists only for  $\alpha \in (0, \alpha_0)$  for some  $\alpha_0 > 0$ . There is some numerical difficulty in performing continuation of multi-pulse solutions to determine the precise value of  $\alpha_0$  owing to the pulses separating with ever increasing speed. Figure 3 shows how this separation varies with  $\alpha$  for the two-pulse solution which strongly suggest that the critical value is  $\alpha_0 = 1$ . Moreover, Figure 3(b) shows the separation to scale like  $1/(1 - \sqrt{\alpha})$  as  $\alpha \rightarrow 1$ . Figures 4 and 5 show in more detail what happens to the two-pulse and three-pulse solutions as  $\alpha \rightarrow 1$ . Solutions on the other branches we computed are observed to behave similarly in this limit. Each multi-pulsed solution retains its amplitude but consists of several infinitely-far separated copies of the primary (up to reflection through  $w = 0$ ).

We tried using numerical shooting at  $\alpha = 1.5$  and could find no solutions other than the primary. We therefore conjecture that there are no multiple pulsed solutions for  $\alpha > 1$ .

The other limit  $\alpha \rightarrow 0$  appears more intriguing. Here, for each of the six branches including the primary, the first harmonic  $w$  can be seen to decay uniformly to zero. The second harmonic  $v$ , however, remains  $\mathcal{O}(1)$  but the scale of  $t$  over which the solution is large appears to tend to infinity. Figure 6 shows all six solutions at  $\alpha = 0.01$ . Note that there is far less qualitative distinction in the profiles of the  $v$  components than there is at  $\alpha = 0.5$ . Note further from Figures 4(b) and 5(b) that at  $\alpha = 0.001$ , the  $w$ -components of the simplest two multiple pulsed solutions appear to be uni-modal.

## 4 Discussion

Aside from issues of practical application of our results, several open analytical questions remain.

Firstly, an analysis of the semi-simple resonant eigenvalues scenario in Hamiltonian reversible systems, could be used to explain the numerically observed bifurcation at  $\alpha = 1$ . This topic is currently

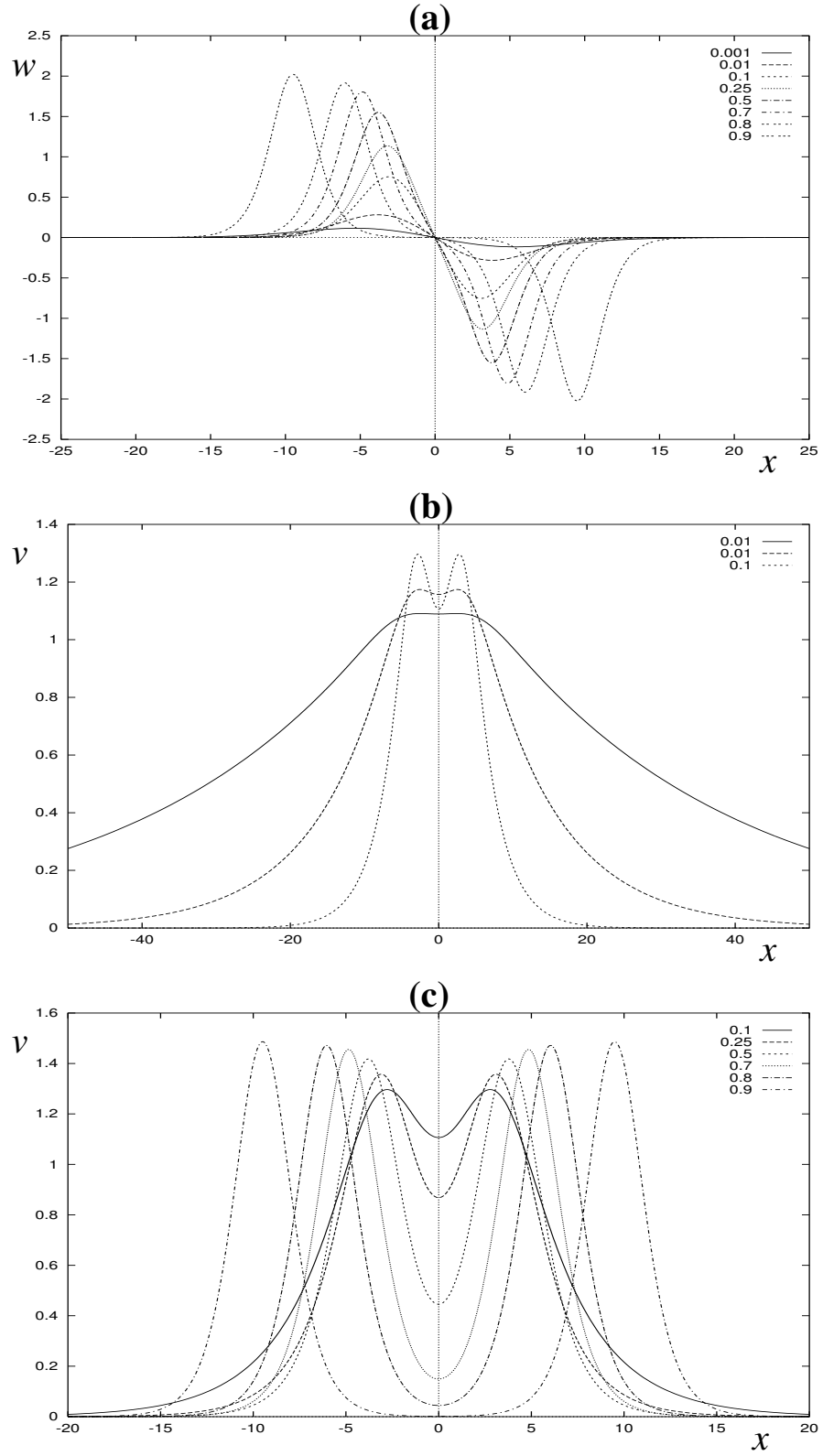


Figure 4: Solutions on the branch b of 2, for the  $\alpha$  values indicated in the key

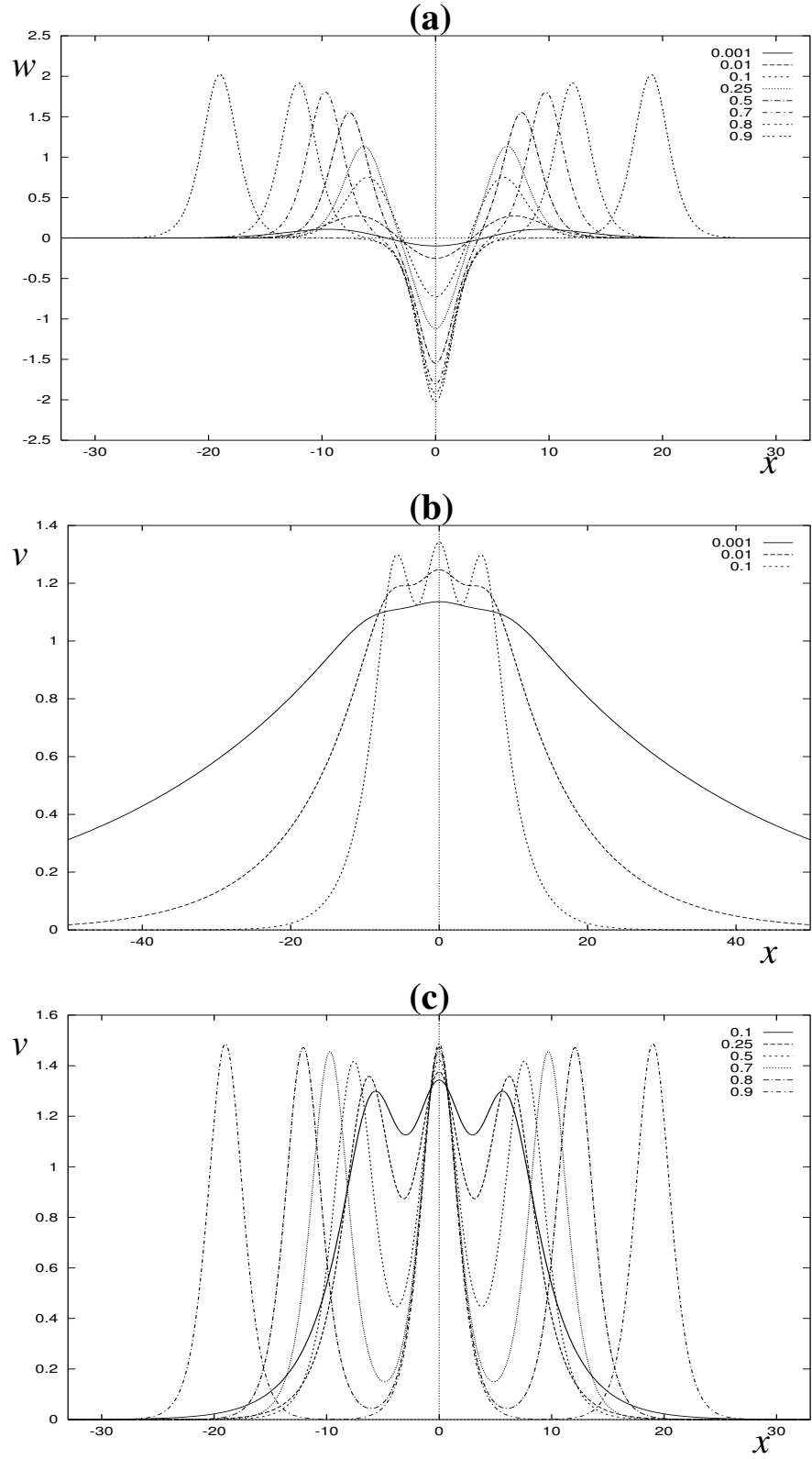


Figure 5: Solutions on the branch c of 2, for the  $\alpha$  values indicated in the key



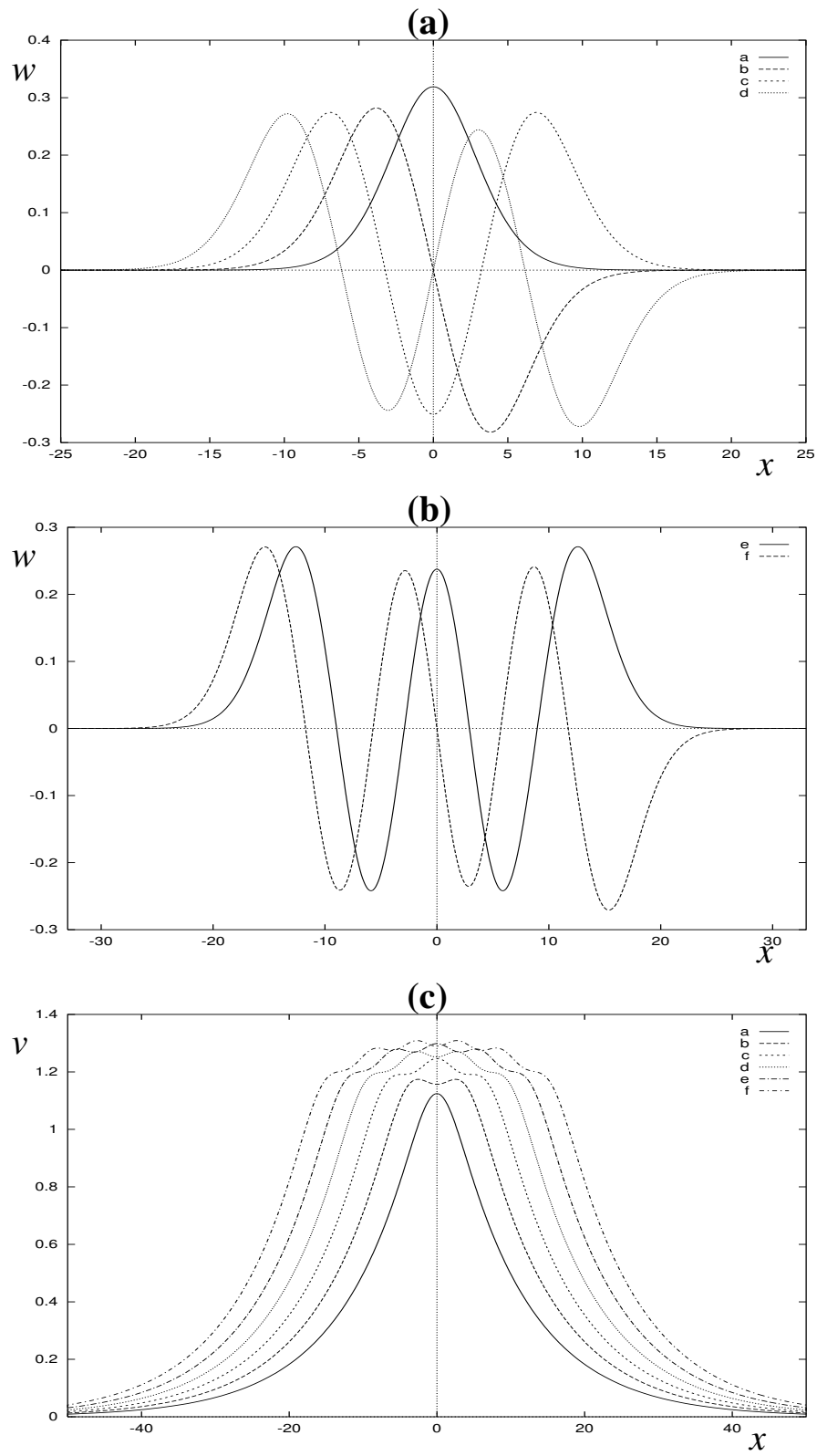


Figure 6: Six solutions at  $\alpha = 0.01$ .

being investigated by the first author using a method developed by B. Sandstede [17] from the ideas of X.-B. Lin. Preliminary analyses suggest that the conjectures made in section 3 can be verified analytically. In particular, in the vicinity of  $\alpha = 1$ , there exist only ‘up-down’ multiple-pulses for  $\alpha < 1$ , which resemble several copies of the  $(\pm \hat{W}(t), \hat{V}(t))$  primary pair; and for the case  $\alpha > 1$ , it appears that no multiple-pulses can be found via the Lin-Sandstede method. It would also seem reasonable to conjecture uniqueness of homoclinic solutions for all  $\alpha > 1$ .

A detailed analytic investigation of the limit  $\alpha \rightarrow 0$  is also pressing. Perhaps singular perturbation theory could shed more light on the numerical observations.

Finally, it would be interesting to see whether the instability of the multi-pulses observed numerically by Haelterman *et al.* [4] for the full PDEs may be proved analytically. A local analysis near  $\alpha = 1$  would appear possible. Also, the numerics in [4] suggest a threshold ( $\alpha = 0.5$  in the present notation) between which two alternative form of instability are observed for the two-pulse solution. For  $\alpha$ -values beyond the threshold, a slightly perturbed initial condition results in the two pulses separating and moving apart with different speeds. For sub-critical  $\alpha$ -values the two-pulse develops a symmetry-breaking instability mechanism where all the energy is transferred into a single pulse. It would be interesting to see if linearised analysis of the PDE could detect these two modes of instability analytically.

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